FILTER-REGULAR SEQUENCES, ALMOST COMPLETE INTERSECTIONS AND STANLEY'S CONJECTURE

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Abstract. Let K be a field and I a monomial ideal of the polynomial ring $S = K[x_1, \dots, x_n]$ generated by monomials $u_1, u_2, \dots, u_t$. We show that S/I is pretty clean if either: 1) $u_1, u_2, \dots, u_t$ is a filter-regular sequence, 2) $u_1, u_2, \dots, u_t$ is a d-sequence; or 3) I is almost complete intersection. In particular, in each of these cases, S/I is sequentially Cohen-Macaulay and both Stanley's and h-regularity conjectures, on Stanley decompositions, hold for S/I. Also, we prove that if I is the Stanley-Reisner ideal of a locally complete intersection simplicial complex on $[n]$, then Stanley's conjecture holds for S/I.

1. Introduction

Throughout, let K be a field and I a monomial ideal of the polynomial ring $S = K[x_1, \dots, x_n]$. A decomposition of S/I as direct sum of K-vector spaces of the form $\mathcal{D} : S/I = \bigoplus_{i=1}^r u_i K[Z_i]$, where u_i is a monomial in S and $Z_i \subseteq \{x_1, \dots, x_n\}$, is called a *Stanley decomposition* of S/I. Stanley conjectured [St] that there always exists a Stanley decomposition $\mathcal{D}$ of S/I such that each Z_i has at least $\operatorname{depth} S/I$ elements. This conjecture is known as Stanley's conjecture. Recently, this conjecture was extensively examined by several authors; see e.g. [A1], [A2], [HP], [HSY], [P], [R], [S4] and [S3]. On the other hand, the present third author [S3] conjectured that there always exists a Stanley decomposition $\mathcal{D}$ of S/I such that degree of each u_i is at most $\operatorname{reg} S/I$. We refer to this conjecture as h-regularity conjecture. It is known that for square-free monomial ideals, these two conjectures are equivalent. Our main aim in this paper is to determine some classes of monomial ideals that these conjectures are true for them.

Let R be a multigraded Noetherian ring and M a finitely generated multigraded R-module. A basic fact in commutative algebra says that there exists a finite filtration

$$\mathcal{F} : 0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

of multigraded submodules of M such that there are multigraded isomorphisms $M_i/M_{i-1} \cong R/\mathfrak{p}_i(-a_i)$ for some $a_i \in \mathbb{Z}^n$ and some multigraded prime ideals $\mathfrak{p}_i$ of R. Such a filtration of M is called a (multigraded) prime filtration. The set of prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ which define the cyclic quotients of $\mathcal{F}$ will be denoted by $\operatorname{Supp} \mathcal{F}$. It is known (and easy to see) that

$$\operatorname{Ass}_R M \subseteq \operatorname{Supp} \mathcal{F} \subseteq \operatorname{Supp}_R M.$$

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Let $\operatorname{Min} M$ denote the set of minimal prime ideals of $\operatorname{Supp}_R M$. Dress [D] called a prime filtration $\mathcal{F}$ of M *clean* if $\operatorname{Supp} \mathcal{F} = \operatorname{Min} M$. Pretty clean filtrations were defined as a generalization of clean filtrations by Herzog and Popescu [HP]. A prime filtration $\mathcal{F}$ is called *pretty clean* if for all $i < j$ for which $\mathfrak{p}_i \subseteq \mathfrak{p}_j$, it follows that $\mathfrak{p}_i = \mathfrak{p}_j$. If $\mathcal{F}$ is a pretty clean filtration of M, then $\operatorname{Supp} \mathcal{F} = \operatorname{Ass}_R M$; see [HP, Corollary 3.4]. The converse is not true in general as shown by some examples in [HP] and [S4]. The prime filtration $\mathcal{F}$ of M is called *almost clean* if $\operatorname{Supp} \mathcal{F} = \operatorname{Ass}_R M$. The R-module M is called *clean* (resp. *pretty clean* or *almost clean*) if it admits a clean (resp. pretty clean or almost clean) filtration. Obviously, cleanness implies pretty cleanness and pretty cleanness implies almost cleanness. When I is square-free, one has $\operatorname{Ass}_S S/I = \operatorname{Min} S/I$, and so these three concepts coincide for R/I. In this paper, we always consider the ring S with its standard multigrading. So, an ideal J of S is multigraded if and only if J is a monomial ideal. Pretty clean modules of the form S/I have very nice properties. If S/I is pretty clean, then S/I is sequentially Cohen-Macaulay and

$$\operatorname{depth} S/I = \min\{\dim S/\mathfrak{p} | \mathfrak{p} \in \operatorname{Ass}_S S/I\};$$

see [S1] for an easy proof. If S/I is pretty clean, then [HP, Theorm 6.5] asserts that Stanley's conjecture holds for S/I. (In fact, this conjecture is true even under the assumption of S/I simply being almost clean; see [S4, Proposition 2.2].) Also if S/I is pretty clean, then by [S3, Theorem 4.7] h-regularity conjecture holds for S/I.

This paper is organized as follows. In the second section, for a multigraded finitely generated S-module M and a multigraded Artinian submodule A of M, we show that M is pretty clean if and only if M/A is pretty clean. Let $u_1, \dots, u_r$ be monomials in S. If $u_1, \dots, u_r$ is a regular sequence on S/I, then by [R, Theorem 2.1] S/I is pretty clean if and only if $S/(I, u_1, \dots, u_r)$ is pretty clean. We show that the same assertion is also true for cleanness and almost cleanness. Also, we prove that if $u_1, \dots, u_r$ is a filter-regular sequence on S/I, then S/I is pretty clean if and only if $S/(I, u_1, \dots, u_r)$ is pretty clean. Next, we show that if $u_1, \dots, u_r$ forms a filter-regular sequence on S/I, then Stanley's conjecture is true for S/I if and only if it is true for $S/(I, u_1, \dots, u_r)$. Assume that $u_1, \dots, u_r$ is a minimal set of generators for an ideal J of S. We prove that if either $u_1, \dots, u_r$ is a d-sequence, proper sequence or strong s-sequence (with respect to the reverse lexicographic order), then S/J is pretty clean.

In the third section, we prove that if the monomial ideal I is either almost complete intersection or it can be generated by less than four monomials, then S/I is pretty clean. Also, we show that if I is the Stanley-Reisner ideal of a locally complete intersection simplicial complex on $[n]$, then S/I satisfies Stanley's conjecture. As a conclusion to our results, we can deduce that both Stanley's and h-regularity conjectures hold for S/I and S/I is sequentially Cohen-Macaulay if either:

i) I can be generated by a filter-regular sequence of monomials,
ii) I can be generated by a d-sequence of monomials,
iii) I is almost complete intersection; or
iv) I can be generated by less than four monomials.
v) I is the Stanley-Reisner ideal of a connected simplicial complex on $[n]$ which is locally complete intersection.

2. Filter-regular sequences and pretty cleanness

In this section, we investigate pretty cleanness in conjunction with filter-regular sequences.

Lemma 2.1. *Let R be a commutative Noetherian ring, M an R-module and A an Artinian submodule of M. Then*

$$\operatorname{Ass}_R M = \operatorname{Ass}_R A \cup \operatorname{Ass}_R M/A.$$

Proof. It is well-known that

$$\operatorname{Ass}_R A \subseteq \operatorname{Ass}_R M \subseteq \operatorname{Ass}_R A \cup \operatorname{Ass}_R M/A.$$

On the other hand, [BSS, Lemma 2.2] yields that

$$\operatorname{Ass}_R M/A \subseteq \operatorname{Ass}_R M \cup \operatorname{Supp}_R A.$$

But A is Artinian, and so $\operatorname{Supp}_R A = \operatorname{Ass}_R A$. This implies our desired equality. □

Lemma 2.2. *Let R be a multigraded Noetherian ring, M a multigraded finitely generated R-module and A a multigraded Artinian submodule of M. If M/A is pretty clean (resp. almost clean), then M is pretty clean (resp. almost clean) too.*

Proof. Since A is an Artinian R-module, one has

$$\operatorname{Min} A = \operatorname{Ass}_R A = \operatorname{Supp}_R A \subseteq \operatorname{Max} R.$$

So obviously, if M/A is pretty clean, then M is pretty clean too. Also, by Lemma 2.1, almost cleanness of M/A implies almost cleanness of M. □

We denote the maximal monomial ideal $(x_1, \ldots, x_n)$ of the ring $S = K[x_1, \ldots, x_n]$ by $\mathfrak{m}$. For a S-module M, $H^i_{\mathfrak{m}}(M)$ denotes ith local cohomology module of M with respect to $\mathfrak{m}$. If M is a multigraded finitely generated S-module, then $H^i_{\mathfrak{m}}(M)$ is a multigraded Artinian S-module for all i.

Example 2.3. Lemma 2.2 is not true for the cleanness. To this end, let $S = K[x, y]$ and $I = (x^2, xy)$. Set $M := S/I$ and $A := H^0_{\mathfrak{m}}(M)$. Clearly $A =< x > /I$, and so $M/A \cong S/ < x >$. It is easy to see that M/A is clean while M is not clean.

Proposition 2.4. *Let M be a multigraded finitely generated S-module and A a multigraded Artinian submodule of M. Then M is pretty clean if and only if M/A is pretty clean.*

Proof. In view of Lemma 2.2, it remains to show that if M is pretty clean, then M/A is pretty clean. Let

$$\mathcal{F} : 0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

be a pretty clean filtration of M. First, by induction on $t := \ell_R(H^0_{\mathfrak{m}}(M))$, we show that $M/H^0_{\mathfrak{m}}(M)$ is pretty clean. For $t = 0$, there is nothing to prove. Now, assume that $t > 0$ and the claim holds for $t - 1$. Then $H^0_{\mathfrak{m}}(M) \neq 0$, and so $\mathfrak{m} \in \operatorname{Ass}_S M = \operatorname{Supp} \mathcal{F}$. Since the filtration $\mathcal{F}$ is pretty clean and $\operatorname{Ann}_R M_1 \subseteq \mathfrak{m}$, it follows that $M_1 \cong S/\mathfrak{m}$, and so $(M_1 :_M \mathfrak{m}^\infty) = H^0_{\mathfrak{m}}(M)$. Then, one has

$$H^0_{\mathfrak{m}}(\frac{M}{M_1}) = \frac{M_1 :_M \mathfrak{m}^\infty}{M_1} = \frac{H^0_{\mathfrak{m}}(M)}{M_1},$$

and so

$$\ell_R(H^0_{\mathfrak{m}}(\frac{M}{M_1})) = \ell_R(H^0_{\mathfrak{m}}(M)) - \ell_R(M_1) = t-1.$$

Obviously, M/M_1 is pretty clean, and so by the induction hypothesis, $\frac{M/M_1}{H^0_{\mathfrak{m}}(M/M_1)}$ is pretty clean. But,

$$\frac{\frac{M}{M_1}}{H^0_{\mathfrak{m}}(\frac{M}{M_1})} = \frac{\frac{M}{M_1}}{\frac{H^0_{\mathfrak{m}}(M)}{M_1}} \cong \frac{M}{H^0_{\mathfrak{m}}(M)},$$

and hence $M/H^0_{\mathfrak{m}}(M)$ is pretty clean.

Since A is a multigraded Artinian submodule of M, one has $A \subseteq H^0_{\mathfrak{m}}(M)$. From the first part of the proof, we conclude that $\frac{M/A}{H^0_{\mathfrak{m}}(M)/A}$ is pretty clean. But $H^0_{\mathfrak{m}}(M)/A$ is a multigraded Artinian submodule of M/A, and so Lemma 2.2 implies that M/A is pretty clean. □

In what follows, we recall some needed notation and facts about monomial ideals. For each subset H of S, let $\operatorname{Mon} H$ denote the set of all monomials in H. For any monomial ideal I of S, there is a unique minimal generating set $\mathrm{G}(I)$ of I. Clearly, $\mathrm{G}(I)$ is consisting of finitely many monomials and there is no divisibility among different elements of $\mathrm{G}(I)$. Also for any non-empty subset T of $\operatorname{Mon} S$, set $\mathrm{G}(T) := \mathrm{G}(< T >)$. Clearly, $\mathrm{G}(< T >)$ is a finite subset of T. A monomial ideal of S is irreducible if and only if it is of the form $(x_{i_1}^{a_1}, \dots, x_{i_t}^{a_t})$, where $a_i \in \mathbb{N}$ for all i; see [HH, Corollary 1.3.2]. Moreover, $(x_{i_1}^{a_1}, \dots, x_{i_t}^{a_t})$ is $(x_{i_1}, \dots, x_{i_t})$-primary and each monomial ideal can be written as a finite intersection of irreducible monomial ideals. Let I be a monomial ideal of S and $\mathcal{P} : I = \bigcap_{i=1}^r Q_i$ a primary decomposition of I such that each Q_i is an irreducible monomial ideal of S. We use notion $T_i(\mathcal{P})$ for $\mathrm{G}(\operatorname{Mon}(\cap_{j=1}^{i-1} Q_j \backslash Q_i))$. Notice that

$$T_1(\mathcal{P}) = \mathrm{G}(\operatorname{Mon}(S \setminus Q_1)) = \{1\}.$$

For proving our first theorem, we shall need the following lemma.

Lemma 2.5. [S2, Corollary 2.7] *Let I be a monomial ideal of S. The following conditions are equivalent:*

a) *S/I is clean (resp. pretty clean or almost clean).*
b) *There exists a primary decomposition $\mathcal{P} : I = \bigcap_{j=1}^r Q_j$ of I, where each Q_j is an irreducible $\mathfrak{p}_j$-primary monomial ideal, such that*
   i) $\operatorname{ht} \mathfrak{p}_j \leq \operatorname{ht} \mathfrak{p}_{j+1}$ *for all j and* $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\} = \operatorname{Min} S/I$,
   (resp. $\operatorname{ht} \mathfrak{p}_j \leq \operatorname{ht} \mathfrak{p}_{j+1}$ *for all j or* $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\} = \operatorname{Ass}_S S/I$*) and*
   ii) *$T_j(\mathcal{P})$ is a singleton for all $1 \leq j \leq r$.*

Next, we generalize [R, Theorem 2.1].

Theorem 2.6. *Let I be a monomial ideal of S and $u_1, \dots, u_c \in \operatorname{Mon} S$ a regular sequence on S/I. Then S/I is clean (resp. pretty clean or almost clean) if and only if $S/(I, u_1, \dots, u_c)$ is clean (resp. pretty clean or almost clean).*

Proof. By induction on c, it is enough to prove the case $c = 1$. Let $u \in \operatorname{Mon} S$ be a non zero-divisor on S/I. Without loss of generality, we may and do assume that for some integer $0 \leq t < n$, the only variables that divide u are $x_{t+1}, \dots, x_n$. Then $u = \prod_{i=t+1}^n x_i^{a_i}$ for some natural integers $a_1, \dots, a_n$ and $I = JS$ for some monomial ideal J of $S' := K[x_1, \dots, x_t]$.

First, we show that if S/I is clean (resp. pretty clean or almost clean), then $S/(I,u)$ is clean (resp. pretty clean or almost clean). Let $\mathcal{P} : I = \cap_{i=1}^{r} Q_i$ be a primary decomposition of I which satisfies the condition b) in Lemma 2.5. Let $1 \leq e \leq r$. Since

$$\mathrm{Ass}_S\, S/I = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$$

and $\mathrm{Ass}_S\, S/Q_e = \{\mathfrak{p}_e\}$, it turns out that u is also a non zero-divisor on S/Q_e. Hence $Q_e = q_e S$ for some irreducible monomial ideal q_e of S'. Obviously,

$$\mathcal{P}' : (I,u) = \big(\cap_{i=t+1}^{n} (Q_1, x_i^{a_i})\big) \cap \big(\cap_{i=t+1}^{n} (Q_2, x_i^{a_i})\big) \cap \ldots \cap \big(\cap_{i=t+1}^{n} (Q_r, x_i^{a_i})\big)$$

is a primary decomposition of (I,u) and each $(Q_i, x_j^{a_j})$ is an irreducible $(\mathfrak{p}_i, x_j)$-primary monomial ideal. We are going to show that the condition b) in Lemma 2.5 holds for $\mathcal{P}'$. Clearly, $T_1(\mathcal{P}')$ is a singleton. For each $t+2 \leq i \leq n$, we have

$$\mathrm{G}(\mathrm{Mon}(\cap_{j=t+1}^{i-1}(Q_1, x_j^{a_j}) \setminus (Q_1, x_i^{a_i}))) = \mathrm{G}(\mathrm{Mon}((Q_1, \prod_{j=t+1}^{i-1} x_j^{a_j}) \setminus (Q_1, x_i^{a_i}))) = \{\prod_{j=t+1}^{i-1} x_j^{a_j}\}.$$

Let $2 \leq i \leq r$, $t+1 \leq h \leq n$ and assume that $T_i(\mathcal{P}) = \{v\}$. Since

$$((\cap_{j=1}^{i-1}\cap_{k=t+1}^{n}(Q_j, x_k^{a_k}))\cap(\cap_{l=t+1}^{h-1}(Q_i, x_l^{a_l})))\setminus(Q_i, x_h^{a_h}) = ((\cap_{j=1}^{i-1}(Q_j, \prod_{k=t+1}^{n} x_k^{a_k}))\cap(Q_i, \prod_{l=t+1}^{h-1} x_l^{a_l}))\setminus(Q_i, x_h^{a_h}),$$

one has

$$\mathrm{G}(\mathrm{Mon}(((\cap_{j=1}^{i-1} \cap_{k=t+1}^{n} (Q_j, x_k^{a_k})) \cap (\cap_{l=t+1}^{h-1}(Q_i, x_l^{a_l}))) \setminus (Q_i, x_h^{a_h}))) = \{v \prod_{l=t+1}^{h-1} x_l^{a_l}\}.$$

So, $T_i(\mathcal{P}')$ is a singleton for all i. On the other hand, we can easily deduce that

$$\mathrm{Ass}_S \frac{S}{(I,u)} = \{(\mathfrak{p}, x_k) | \mathfrak{p} \in \mathrm{Ass}_S \frac{S}{I} \text{ and } t+1 \leq k \leq n\} \quad (*),$$

$$\mathrm{Min} \frac{S}{(I,u)} = \{(\mathfrak{p}, x_k) | \mathfrak{p} \in \mathrm{Min} \frac{S}{I} \text{ and } t+1 \leq k \leq n\} \quad (\dagger)$$

and $\mathrm{ht}(\mathfrak{p}, x_k) = \mathrm{ht}\,\mathfrak{p} + 1$ $(\ddagger)$ for all $\mathfrak{p} \in \mathrm{Ass}_S\, S/I$ and all $t+1 \leq k \leq n$. Hence $\mathcal{P}'$ satisfies the condition b) in Lemma 2.5.

Conversely, let $S/(I,u)$ be clean (resp. pretty clean or almost clean). So, (I,u) has a primary decomposition $\mathcal{P}$ which satisfies the condition b) in Lemma 2.5. From $(*)$, we can conclude that $\mathcal{P}$ has the form

$$\mathcal{P} : (I,u) = (Q_1, x_{j_1}^{h_{j_1}}) \cap (Q_2, x_{j_2}^{h_{j_2}}) \cap \ldots \cap (Q_s, x_{j_s}^{h_{j_s}}),$$

where for each $1 \leq i \leq s$, $Q_i = q_i S$ for some irreducible monomial ideal q_i of S', $\sqrt{Q_i} \in \mathrm{Ass}_S\, S/I$ and $j_i \in \{t+1, \ldots, n\}$. It follows that $I = \cap_{i=1}^{s} Q_i$ is a primary decomposition of I. By deleting unneeded components, we get a primary decomposition

$$\mathcal{P}' : I = Q_{i_1} \cap Q_{i_2} \cap \ldots \cap Q_{i_l}$$

such that $i_1 < i_2 < \cdots < i_l$ and for each $1 \leq j \leq l$, $\cap_{k<j} Q_{i_k} \nsubseteq Q_{i_j}$ and $\cap_{k<j} Q_{i_k} = \cap_{m<i_j} Q_m$. We intend to show that $\mathcal{P}'$ satisfies the condition b) in Lemma 2.5. Since

$$\mathrm{Ass}_S\, S/I = \{\sqrt{Q_{i_1}}, \sqrt{Q_{i_2}}, \ldots, \sqrt{Q_{i_l}}\},$$

in view of $(*)$, $(\dagger)$ and $(\ddagger)$, we only need to indicate that each $T_i(\mathcal{P}')$ is a singleton. Let $1 \leq j \leq l$. Since $\cap_{k<j} Q_{i_k} \nsubseteq Q_{i_j}$, it follows that there exists at least a monomial v in $\mathrm{G}(\cap_{k<j} Q_{i_k}) \setminus Q_{i_j}$. We claim that v is unique. If there exists a monomial $w \neq v$ in $\mathrm{G}(\cap_{k<j} Q_{i_k}) \setminus Q_{i_j}$, then since $\cap_{k<j} Q_{i_k} = \cap_{m<i_j} Q_m$, it turns out that v and w are belonging to $\mathrm{G}(\cap_{m<i_j} Q_m) \setminus Q_{i_j}$. Denote i_j by d. Since $v, w \in S'$, we can conclude that v and w are belonging to

$$\mathrm{G}((Q_1, x_{j_1}^{h_{j_1}}) \cap (Q_2, x_{j_2}^{h_{j_2}}) \cap \ldots \cap (Q_{d-1}, x_{j_{d-1}}^{h_{j_{d-1}}})) \setminus (Q_d, x_{j_d}^{h_{j_d}}).$$

This contradicts the assumption that $T_d(\mathcal{P})$ is a singleton. Therefore, each $T_i(\mathcal{P}')$ is a singleton, as desired. $\square$

As an immediate consequence, we obtain the following result; see [HSY, Proposition 2.2].

Corollary 2.7. *Let $u_1, \ldots, u_t \in \operatorname{Mon} S$ be a regular sequence on S. Then $S/(u_1, \ldots, u_t)$ is clean.*

Definition 2.8. Let M be a multigraded finitely generated S-module and $\mathbf{u} = u_1, \ldots, u_r$ a sequence of non-unite monomials in S. We call $\mathbf{u}$ a *filter-regular sequence* on M if for each $1 \leq i \leq r$

$$u_i \notin \bigcup_{\mathfrak{p} \in \operatorname{Ass}_S \left(\frac{M}{(u_1, \ldots, u_{i-1})M}\right) - \{\mathfrak{m}\}} \mathfrak{p}.$$

Lemma 2.9. *Let M be a multigraded finitely generated S-module. An element $1 \neq u \in \operatorname{Mon} S$ is a filter-regular sequence on M if and only if it is a non zero-devisor on $M/H^0_{\mathfrak{m}}(M)$.*

Proof. Since $H^0_{\mathfrak{m}}(M)$ is Artinian and $H^0_{\mathfrak{m}}(\frac{M}{H^0_{\mathfrak{m}}(M)}) = 0$, Lemma 2.1 yields that

$$\operatorname{Ass}_S(\frac{M}{H^0_{\mathfrak{m}}(M)}) = \operatorname{Ass}_R M - \{\mathfrak{m}\}.$$

Hence, by definition the claim is immediate. $\square$

Theorem 2.10. *Let I be a monomial ideal of S and $u_1, \ldots, u_r \in \operatorname{Mon} S$ a filter-regular sequence on S/I. Then S/I is pretty clean if and only if $S/(I, u_1, \ldots, u_r)$ is pretty clean.*

Proof. By induction on r, it is enough to prove that for a monomial filter-regular sequence u on S/I, S/I is pretty clean if and only if $S/(I, u)$ is pretty clean. For convenience, we set $M := S/I$. By Proposition 2.4, M is pretty clean if and only if $M/H^0_{\mathfrak{m}}(M)$ is pretty clean. By Lemma 2.9, u is a non zero-divisor on $M/H^0_{\mathfrak{m}}(M)$. Hence, in view of the isomorphism

$$\frac{\frac{M}{H^0_{\mathfrak{m}}(M)}}{u(\frac{M}{H^0_{\mathfrak{m}}(M)})} \cong \frac{M}{uM + H^0_{\mathfrak{m}}(M)},$$

Theorem 2.6 yields that $M/H^0_{\mathfrak{m}}(M)$ is pretty clean if and only if $\frac{M}{uM+H^0_{\mathfrak{m}}(M)}$ is pretty clean. On the other hand, as $\frac{uM+H^0_{\mathfrak{m}}(M)}{uM}$ is a multigraded Artinian submodule of M/uM, by Proposition 2.4 and the isomorphism

$$\frac{M}{uM + H^0_{\mathfrak{m}}(M)} \cong \frac{\frac{M}{uM}}{\frac{uM+H^0_{\mathfrak{m}}(M)}{uM}},$$

it turns out that $\frac{M}{uM+H^0_{\mathfrak{m}}(M)}$ is pretty clean if and only if M/uM is pretty clean. Therefore, M is pretty clean if and only if M/uM is pretty clean. $\square$

Corollary 2.11. *Let monomials $u_1, \dots, u_r$ be a filter-regular sequence on S. Then $S/(u_1, \dots, u_r)$ is pretty clean.*

Lemma 2.12. *Let M be a multigraded finitely generated S-module and $u_1, \dots, u_r \in \operatorname{Mon} S$ a filter-regular sequence on M. If $\mathfrak{m} \in \operatorname{Ass}_S M$, then $\mathfrak{m} \in \operatorname{Ass}_S(M/(u_1, \dots, u_r)M)$.*

Proof. By induction on r, it is enough to prove that if u is a monomial filter-regular sequence on M and $\mathfrak{m} \in \operatorname{Ass}_S M$, then $\mathfrak{m} \in \operatorname{Ass}_S M/uM$. Since $\mathfrak{m} \in \operatorname{Ass}_S M$, there exists $0 \neq x \in M$ such that $\mathfrak{m} = 0 :_S x$. Then, there exists a non-negative integer t such that $x \in u^t M \setminus u^{t+1} M$. Hence $x = u^t y$ for some $y \in M \setminus uM$. Clearly, $0 :_S y \subset S$. Let $\mathfrak{p} \subset \mathfrak{m}$ be a prime ideal of S containing $0 :_S y$. Since u is a filter-regular sequence on M and $\mathfrak{p} \neq \mathfrak{m}$, it follows that $\frac{u}{1} \in S_{\mathfrak{p}}$ is $M_{\mathfrak{p}}$-regular. Hence

$$(0 :_S x)_{\mathfrak{p}} = 0 :_{S_{\mathfrak{p}}} \frac{u^t}{1}\frac{y}{1} = 0 :_{S_{\mathfrak{p}}} \frac{y}{1} = (0 :_S y)_{\mathfrak{p}} \subseteq \mathfrak{p} S_{\mathfrak{p}},$$

and so

$$(0 :_S x) \subseteq (0 :_S x)_{\mathfrak{p}} \cap S \subseteq \mathfrak{p} S_{\mathfrak{p}} \cap S = \mathfrak{p}.$$

This is a contradiction, and so $\mathfrak{m}$ is the unique prime ideal of S containing $(0 :_S y)$. So,

$$\mathfrak{m} = \sqrt{(0 :_S y)} \subseteq \sqrt{(0 :_S y + uM)} \subset S.$$

Therefore, $\sqrt{(0 :_S y + uM)} = \mathfrak{m}$, and so $\mathfrak{m} \in \operatorname{Ass}_S M/uM$. $\square$

Theorem 2.13. *Let I be a monomial ideal of S and $u_1, \dots, u_r \in \operatorname{Mon} S$ a filter-regular sequence on S/I. Then Stanley's conjecture holds for S/I if and only if it holds for $S/(I, u_1, \dots, u_r)$.*

Proof. By induction on r, it is enough to prove that if u is a monomial filter-regular sequence on S/I, then Stanley's conjecture holds for S/I if and only if it holds for $S/(I, u)$. First, assume that $\mathfrak{m} \in \operatorname{Ass}_S S/I$. Then $\operatorname{depth} S/I = 0$ and by Lemma 2.12, $\mathfrak{m} \in \operatorname{Ass}_S S/(I, u)$. So, $\operatorname{depth} S/(I, u) = 0$. Hence the claim is immediate in this case. Now, assume that $\mathfrak{m} \notin \operatorname{Ass}_S S/I$. Then u is a non zero-divisor on S/I, and so by [R, Theorem 1.1], Stanley's conjecture holds for S/I if and only if it holds for $S/(I, u)$. $\square$

Definition 2.14. Let R ba a commutative Noetherian ring, M a finitely generated R-module and $f_1, \dots, f_t \in R$.

i) $f_1, \dots, f_t$ is called a *d-sequence* on M if $f_1, \dots, f_t$ is a minimal generating set of the ideal $(f_1, \dots, f_t)$ and $(f_1, \dots, f_i)M :_M f_{i+1} f_k = (f_1, \dots, f_i)M :_M f_k$ for all $0 \leq i < t$ and all $k \geq i+1$. A d-sequence on R is simply called a d-sequence.

ii) $f_1, \dots, f_t$ is called a *proper sequence* if $f_{i+1} H_j(f_1, \dots, f_i; R) = 0$ for all $0 \leq i < t$ and all $j > 0$. Here $H_j(f_1, \dots, f_i; R)$ denotes the jth Koszul homology of R with respect to $f_1, \dots, f_i$.

iii) Let $M = (g_1, \dots, g_t)$ and $(a_{ij})_{s \times t}$ be its relation matrix. Then the symmetric algebra of M is defined by $\operatorname{Sym} M := R[y_1, \dots, y_t]/J$, where $J = (\sum_{j=1}^t a_{1j} y_j, \dots, \sum_{j=1}^t a_{sj} y_j)$. Let $<$ be a monomial order on the monomials in $y_1, \dots, y_n$ with the property $y_1 < \cdots < y_n$. Set $I_i := (g_1, \dots, g_{i-1}) :_S g_i$. Then $(I_1 y_1, \dots, I_t y_t) \subseteq \operatorname{in}_< J$. We call $g_1, \dots, g_t$ a *s-sequence* (with respect to $<$) if $(I_1 y_1, \dots, I_t y_t) = \operatorname{in}_< J$. If in addition $I_1 \subseteq \cdots \subseteq I_t$, then $g_1, \dots, g_t$ is called a *strong s-sequence.*

Definition 2.15. Let I be a (not necessarily square-free) monomial ideal of S with $\mathrm{G}(I) = \{u_1, ..., u_m\}$. A monomial u_t is called a leaf of $\mathrm{G}(I)$ if u_t is the only element in $\mathrm{G}(I)$ or there exists a $j \neq t$ such that $\gcd(u_t, u_i) | \gcd(u_t, u_j)$ for all $i \neq t$. In this case, u_j is called a branch of u_t . We say that I is a monomial ideal of forest type if any non-empty subset of $\mathrm{G}(I)$ has a leaf.

[SZ, Theorem 1.5] yields that if I is a monomial ideal of forest type, then S/I is pretty clean.

Lemma 2.16. *Let $u_1, \dots, u_t$ be a sequence of monomials with the following properties:*

i) *there is no $i \neq j$ such that $u_i | u_j$; and*
ii) $\gcd(u_i, u_j) | u_k$ *for all* $1 \leq i < j < k \leq t$.

Then $I = (u_1, \dots, u_t)$ is of forest type, and so S/I is pretty clean.

Proof. For any non-empty subset $A = \{u_{n_1}, \dots, u_{n_s}\}$ of $\{u_1, \dots, u_t\}$, we may and do assume that $n_1 < n_2 < \cdots < n_s$. Then obviously the first element of A is a leaf and the last element of A is a branch for that leaf. So, I is of forest type. Then [SZ, Theorem 1.5] implies that S/I is pretty clean. □

Proposition 2.17. *Let I be a monomial ideal of S with $\mathrm{G}(I) = \{u_1, \dots, u_t\}$. If $u_1, \dots, u_t$ is a d-sequence, proper sequence or strong s-sequence (with respect to the reverse lexicographic order), then S/I is pretty clean.*

Proof. By [HRT, Corollaries 3.3 and 3.4] any d-sequence is a strong s-sequence with respect to the reverse lexicographic order and $u_1, \dots, u_t$ is a proper sequence if and only if it is a strong s-sequence with respect to the reverse lexicographic order. So, by the hypothesis and [T, Theorem 3.1], there is no $i \neq j$ such that $u_i | u_j$ and $\gcd(u_i, u_j) | u_k$ for all $1 \leq i < j < k \leq t$. Hence, by Lemma 2.16, S/I is pretty clean. □

Let I be a monomial ideal of S and u a monomial which is a d-sequence on S/I. The following example shows that it may happen that S/I is pretty clean, but $S/(I, u)$ is not.

Example 2.18. Let $I = (x_1x_2, x_2x_3, x_3x_4)$ be a monomial ideal of $S = K[x_1, x_2, x_3, x_4]$. It is easy to see that S/I is pretty clean and x_4x_1 is a d-sequence on S/I. But, by [S4, Example 1.11], we know that $S/(I, x_4x_1) = S/(x_1x_2, x_2x_3, x_3x_4, x_4x_1)$ is not pretty clean.

3. Almost and locally complete intersection monomial ideals

A *simplicial complex* Δ on $[n] := \{1, \dots, n\}$ is a collection of subsets of $[n]$ with the property if $F \in \Delta$, then all subsets of F are also in Δ. Any singleton element of Δ is called a *vertex*. An element of Δ is called a *face* of Δ and the maximal faces of Δ, under inclusion, are called *facets*. We denote by $\mathcal{F}(\Delta)$ the set of all facets of Δ. The *dimension* of a face F is defined as $\dim F = |F| - 1$, where $|F|$ is the number of elements of F. The dimension of the simplicial complex Δ is the maximal dimension of its facets. A simplicial complex Δ is called *pure* if all facets of Δ have the same dimension. We denote the simplicial complex Δ with facets $F_1, \dots, F_t$ by $\Delta = \langle F_1, \dots, F_t \rangle$. According to Björner and Wachs [BW], a simplicial complex Δ is said to be *(non-pure) shellable* if there exists an order $F_1, \dots, F_t$ of the facets of Δ such that for each $2 \leq i \leq t$, $\langle F_1, \dots, F_{i-1} \rangle \cap \langle F_i \rangle$ is a pure $(\dim F_i - 1)$-dimensional simplicial complex. If Δ is a simplicial complex on $[n]$, then the *Stanley-Reisner ideal* of Δ, I_Δ, is the square-free monomial ideal generated by all monomials $x_{i_1}x_{i_2}\dots x_{i_t}$ such that $\{i_1, i_2, \dots, i_t\} \notin \Delta$. The

Stanley-Reisner ring of Δ over the field K is the K-algebra $K[\Delta] := S/I_\Delta$. Any square-free monomial ideal I is the Stanley-Reisner ideal of some simplicial complex Δ on $[n]$. If $\mathcal{F}(\Delta) = \{F_1, \dots, F_t\}$, then $I_\Delta = \bigcap_{i=1}^t \mathfrak{p}_{F_i}$, where $\mathfrak{p}_{F_i} := (x_j : j \notin F_i)$; see [BH, Theorem 5.1.4].

Recall that the *Alexander dual* $\Delta^\vee$ of a simplicial complex Δ is the simplicial complex whose faces are $\{[n]\backslash F | F \notin \Delta\}$. Let I be a square-free monomial ideal of S. We denote by $I^\vee$, the square-free monomial ideal which is generated by all monomials $x_{i_1} \cdots x_{i_k}$, where $(x_{i_1}, \dots, x_{i_k})$ is a minimal prime ideal of I. It is easy to see that for any simplicial complex Δ, one has $I_{\Delta^\vee} = (I_\Delta)^\vee$. A monomial ideal I of S is said to have *linear quotients* if there exists an order $u_1, \dots, u_m$ of $\mathrm{G}(I)$ such that for any $2 \leq i \leq m$, the ideal $(u_1, \dots, u_{i-1}) :_S u_i$ is generated by a subset of the variables.

Lemma 3.1. *Let I be a square-free monomial ideal of S. Then S/I is clean if and only if $I^\vee$ has linear quotients.*

Proof. Dress [D, Theorem on page 53] proved that a simplicial complex Δ is (non-pure) shellable if and only if $K[\Delta]$ is a clean ring. On the other hand, by [HHZ, Theorem 1.4], a simplicial complex Δ is (non-pure) shellable if and only if $I_{\Delta^\vee}$ has linear quotients. Combining these facts, yields our claim. $\square$

Lemma 3.2. *Let I and J be two monomial ideals of S. Assume that $I = uJ$ for some monomial u in S and* $\operatorname{ht} J \geq 2$. *If S/J is pretty clean, then S/I is pretty clean too.*

Proof. With the proof of [S4, Lemma 1.9], the claim is immediate. $\square$

In what follows for a monomial ideal I of S, we denote the number of elements of $\mathrm{G}(I)$ by $\mu(I)$.

Definition 3.3. A monomial ideal I of S is said to be *almost complete intersection* if $\mu(I) = \operatorname{ht} I + 1$.

Lemma 3.4. *Let I be an almost complete intersection square-free monomial ideal of S. Then S/I is clean.*

Proof. The claim is obvious when $\operatorname{ht} I = 0$. Let $\operatorname{ht} I = 1$. Then $I = (u_1, u_2)$ for some monomials u_1 and u_2. We can write I as $I = u(u_1', u_2')$, where $u = \gcd(u_1, u_2)$ and u_1', u_2' are monomials forming a regular sequence on S. So in this case, the claim is immediate by Lemma 3.2 and Corollary 2.7. Now, assume that $h := \operatorname{ht} I \geq 2$. By [KTY, Theorem 4.4] I can be written in one of the following forms, where $A_1, A_2, \dots, B_1, B_2, \dots$ are non-trivial square-free monomials no two of which have any common factor, and p, p' are integers with $2 \leq p \leq h$ and $1 \leq p' \leq h$.

1) $I_1 = (A_1B_1, A_2B_2, \dots, A_pB_p, A_{p+1}, \dots, A_h, B_1B_2 \cdots B_p)$.
2) $I_2 = (A_1B_1, A_2B_2, \dots, A_{p'}B_{p'}, A_{p'+1}, \dots, A_h, A_{h+1}B_1B_2 \cdots B_{p'})$.
3) $I_3 = (B_1B_2, B_1B_3, B_2B_3, A_4, \dots, A_{h+1})$.
4) $I_4 = (A_1B_1B_2, B_1B_3, B_2B_3, A_4, \dots, A_{h+1})$.
5) $I_5 = (A_1B_1B_2, A_2B_1B_3, B_2B_3, A_4, \dots, A_{h+1})$.
6) $I_6 = (A_1B_1B_2, A_2B_1B_3, A_3B_2B_3, A_4, \dots, A_{h+1})$.

Let $I = I_1$. Since no two of $A_1, A_2, \dots, A_p, A_{p+1}, \dots, A_h, B_1, B_2, \dots, B_p$ have any common factor, it turns out that $A_{p+1}, \dots, A_h$ is a regular sequence on $S/(A_1B_1, A_2B_2, \dots, A_pB_p, B_1B_2 \cdots B_p)$. So, in view of Theorem 2.6, we may and do assume that $I = (A_1B_1, A_2B_2, \dots, A_pB_p, B_1B_2 \cdots B_p)$. Next, we are going to show that I is of forest type. Let G be a subset of $\{A_1B_1, A_2B_2, \dots, A_pB_p, B_1B_2 \cdots B_p\}$ with at least

two elements. If $B_1B_2\cdots B_p \notin G$, then any $a \in G$ can be taken as a leaf and any $b \in G$ different from a can be taken as a branch for this leaf. If $B_1B_2\cdots B_p \in G$, then any $a \in G$ different from $B_1B_2\cdots B_p$ can be taken as a leaf and then $B_1B_2\cdots B_p$ is a branch for this leaf. So, I is of forest type. Thus, as I is square-free, [SZ, Theorem 1.5] implies that S/I is clean. By the similar argument, one can see that if $I = I_2$, then S/I is clean. Set

$$J := (C_1B_1B_2, C_2B_1B_3, C_3B_2B_3, A_4, \dots, A_{h+1}),$$

where C_i is either A_i or 1 for each $i = 1, 2, 3$. Since each of I_3, I_4, I_5 and I_6 are the particular cases of the ideal J, we can finish the proof by showing that S/J is clean. Since by the assumption no two of $A_4, \dots, A_{h+1}, B_1, B_2, B_3, C_1, C_2, C_3$ have any common factor, it follows that $A_4, \dots, A_{h+1}$ is a regular sequence on $S/(C_1B_1B_2, C_2B_1B_3, C_3B_2B_3)$. So by Theorem 2.6, we can assume that $J = (C_1B_1B_2, C_2B_1B_3, C_3B_2B_3)$. By Lemma 3.1, it is enough to prove that $J^\vee$ has linear quotients. By the hypothesis, we can set $B_1 = x_1\cdots x_l$, $B_2 = y_1\cdots y_s$, $B_3 = z_1\cdots z_t$, $C_1 = u_1\cdots u_d$, $C_2 = v_1\cdots v_m$ and $C_3 = w_1\cdots w_e$. If $C_i = 1$ for some $i = 1, 2, 3$, then instead of all variables corresponding to C_i, we simply put 1. Now, we may and do assume that

$$S = K[x_1, \dots, x_l, y_1, \dots, y_s, z_1, \dots, z_t, u_1, \dots, u_d, v_1, \dots, v_m, w_1, \dots, w_e].$$

Next, as

$$J = (\prod_{h=1}^{d}\prod_{i=1}^{l}\prod_{j=1}^{s} u_hx_iy_j,\ \prod_{p=1}^{m}\prod_{i=1}^{l}\prod_{k=1}^{t} v_px_iz_k,\ \prod_{q=1}^{e}\prod_{j=1}^{s}\prod_{k=1}^{t} w_qy_jz_k),$$

it is easy to see that

$$J = (\bigcap_{i,j}(x_i, y_j)) \cap (\bigcap_{i,k}(x_i, z_k)) \cap (\bigcap_{i,q}(x_i, w_q)) \cap (\bigcap_{j,k}(y_j, z_k)) \cap (\bigcap_{j,p}(y_j, v_p)) \cap (\bigcap_{k,h}(z_k, u_h)) \cap (\bigcap_{h,p,q}(u_h, v_p, w_q)).$$

Thus

$$\begin{aligned} G(J^\vee) = \{x_iy_j \mid 1 \le i \le l,\ 1 \le j \le s\} \cup \{x_iz_k \mid 1 \le i \le l,\ 1 \le k \le t\} \cup \{x_iw_q \mid 1 \le i \le l,\ 1 \le q \le e\} \\ \cup\{y_jz_k \mid 1 \le j \le s,\ 1 \le k \le t\} \cup \{y_jv_p \mid 1 \le j \le s,\ 1 \le p \le m\} \cup \{z_ku_h \mid 1 \le k \le t,\ 1 \le h \le d\} \\ \cup\{u_hv_pw_q \mid 1 \le h \le d,\ 1 \le p \le m,\ 1 \le q \le e\}. \end{aligned}$$

Let $>$ be the pure lexicographic ordering on $\operatorname{Mon} S$ with

$$x_1 > \cdots > x_l > y_1 > \cdots > y_s > z_1 > \cdots > z_t > u_1 > \cdots > u_d > v_1 > \cdots > v_m > w_1 > \cdots > w_e.$$

If $C_i = 1$ for some $i = 1, 2, 3$, then we delete the variables corresponding to C_i in the above chain. Now, arrange elements of $G(J^\vee) = \{d_1, d_2, \dots, d_g\}$ such that either $\deg d_i$ is less than $\deg d_{i+1}$ or if $\deg d_i = \deg d_{i+1}$, then $d_i > d_{i+1}$. Then, it is straightforward to check that $J^\vee$ has linear quotients. $\square$

Let $u = \prod_{i=1}^{n} x_i^{a_i}$ be a monomial in $S = K[x_1, \dots, x_n]$. Then

$$u^p := \prod_{i=1}^{n}\prod_{j=1}^{a_i} x_{i,j} \in K[x_{1,1}, \dots, x_{1,a_1}, \dots, x_{n,1}, \dots, x_{n,a_n}]$$

is called *polarization* of u. Let I be a monomial ideal of S with $\mathrm{G}(I) = \{u_1, \dots, u_m\}$. Then the ideal $I^p := (u_1^p, \dots, u_m^p)$ of $T := K[x_{i,j}]$ is called *polarization* of I. [S4, Theorem 3.10] implies that S/I is pretty clean if and only if T/I^p is clean.

Theorem 3.5. *Let I be an almost complete intersection monomial ideal of S. Then S/I is pretty clean.*

Proof. From [F, Proposition 2.3], one has $\operatorname{ht} I = \operatorname{ht} I^p$. On the other hand $\mu(I) = \mu(I^p)$, and so I^p is an almost complete intersection square-free monomial ideal of T. Hence, by Lemma 3.4, the ring T/I^p is clean. Now, [S4, Theorem 3.10] implies that S/I is pretty clean, as desired. □

In [C, Theorem 2.3], it is shown that if I is a monomial ideal of S with $\mu(I) \leq 3$, then Stanley's conjecture holds for S/I. The next result extends this fact.

Corollary 3.6. *Let I be a monomial ideal of S. If $\mu(I) \leq 3$, then S/I is pretty clean.*

Proof. Clearly, we may assume that I is non zero. Assume that $\mu(I) = 3$ and $\operatorname{ht} I = 1$. Then $I = uJ$, where u is a monomial in S and J is a monomial ideal of S with $\mu(J) = 3$ and $\operatorname{ht} J \geq 2$. By Lemma 3.2, it is enough to prove that S/J is pretty clean. If $\operatorname{ht} J = 2$, then $\mu(J) = \operatorname{ht} J + 1$, and so by Theorem 3.5, S/J is pretty clean. If $\operatorname{ht} J = 3$, then J is complete intersection, and hence by Corollary 2.7, S/J is pretty clean.

Since $0 < \operatorname{ht} I \leq \mu(I)$, in all other cases, it follows that I is either complete intersection or almost complete intersection. Thus, the proof is completed by Corollary 2.7 and Theorem 3.5. □

Definition 3.7. ([TY, Definition 1.1 and Lemma 1.2]) A simplicial complex Δ on $[n]$ is said to be *locally complete intersection* if $\{\{1\}, \{2\}, \ldots, \{n\}\} \subseteq \Delta$ and $(I_\Delta)_\mathfrak{p}$ is a complete intersection ideal of $S_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Proj} S/I$.

A simplicial complex Δ is said to be *connected* if for any two facets F and G of Δ, there exists a sequence of facets $F = F_0, F_1, \ldots, F_{q-1}, F_q = G$ such that $F_i \cap F_{i+1} \neq \emptyset$ for all $0 \leq i < q$. Also, a simplicial complex Δ on $[n]$ is said to be *n-pointed path* (resp. *n-gon*) if $n \geq 2$ (resp. $n \geq 3$) and, after a suitable change of variables,

$$\mathcal{F}(\Delta) = \{\{i, i+1\} | 1 \leq i < n\}$$

(resp.

$$\mathcal{F}(\Delta) = \{\{i, i+1\} | 1 \leq i < n\} \cup \{\{n, 1\}\}).$$

Clearly, any n-pointed path (resp. n-gon) is one-dimensional and pure.

Lemma 3.8. *Let Δ be a connected simplicial complex on $[n]$ which is locally complete intersection. Then S/I_Δ is clean.*

Proof. If $\dim \Delta = 0$, then $\Delta = \{\emptyset, \{1\}, \{2\}, \ldots, \{n\}\}$, and so Δ is shellable. Hence, the claim is obvious in this case by [D, Theorem on page 53].

If $\dim \Delta = 1$, then by [TY, Proposition 1.11] Δ is either a n-pointed path or a n-gon. Obviously, in each of these cases, Δ is shellable, and so by [D, Theorem on page 53] it turns out that S/I_Δ is clean.

If $\dim \Delta \geq 2$, then [TY, Theorem 1.5] implies that I_Δ is generated by a regular sequence. Thus Corollary 2.7 completes the proof in this case. □

Proposition 3.9. *Let $I \subset S_1 = K[x_1, \ldots, x_m]$, $J \subset S_2 = K[x_{m+1}, \ldots, x_n]$ be two monomial ideals and $S = K[x_1, \ldots, x_m, x_{m+1}, \ldots, x_n]$. Assume that* $\operatorname{depth} S_1/I > 0$ *and* $\operatorname{depth} S_2/J > 0$. *Then Stanley's conjecture holds for $S/(I, J, \{x_i x_j\}_{1 \leq i \leq m, m+1 \leq j \leq n})$.*

Proof. For convenience, we set $Q_1 := (x_1, \dots, x_m)$, $Q_2 := (x_{m+1}, \dots, x_n)$ and $Q := (x_i x_j)_{1 \le i \le m, m+1 \le j \le n}$. So, $Q = Q_1 \cap Q_2$. Since $I \subseteq Q_1$ and $J \subseteq Q_2$, it follows that

$$(I, J, Q) = (I, J, Q_1) \cap (I, J, Q_2) = (J, Q_1) \cap (I, Q_2).$$

By the assumption, we have $(x_1, \dots, x_m) \notin \operatorname{Ass}_{S_1} S_1/I$ and $(x_{m+1}, \dots, x_n) \notin \operatorname{Ass}_{S_2} S_2/J$. Hence

$$(x_1, \dots, x_m, x_{m+1}, \dots, x_n) \notin \operatorname{Ass}_S S/(I, Q_2)$$

and

$$(x_1, \dots, x_m, x_{m+1}, \dots, x_n) \notin \operatorname{Ass}_S S/(J, Q_1),$$

and so

$$\operatorname{depth}(\frac{S}{(J, Q_1)} \oplus \frac{S}{(I, Q_2)}) > 0 = \operatorname{depth}(\frac{S}{Q_1 + Q_2}).$$

Now, in view of the exact sequence

$$0 \to \frac{S}{(J, Q_1) \cap (I, Q_2)} \to \frac{S}{(J, Q_1)} \oplus \frac{S}{(I, Q_2)} \to \frac{S}{Q_1 + Q_2} \to 0,$$

[V, Lemma 1.3.9] implies that

$$\operatorname{depth}(\frac{S}{(I, J, Q)}) = \operatorname{depth}(\frac{S}{(J, Q_1) \cap (I, Q_2)}) = 1.$$

Now the proof is complete, because [C, Theorem 2.1] yields that for any monomial ideals L of S if $\operatorname{depth} S/L \le 1$, then Stanley's conjecture holds for S/L. □

Corollary 3.10. *Let Δ_1 and Δ_2 be two non-empty disjoint simplicial complexes and $\Delta := \Delta_1 \cup \Delta_2$. Then Stanley's conjecture holds for S/I_Δ.*

Proof. For two natural integers $m < n$, we may assume that Δ_1 and Δ_2 are simplicial complexes on $[m]$ and $\{m+1, \dots, n\}$, respectively. Then $K[\Delta_1] = K[x_1, \dots, x_m]/I_{\Delta_1}$ and $K[\Delta_2] = K[x_{m+1}, \dots, x_n]/I_{\Delta_2}$, and so

$$K[\Delta] = K[x_1, \dots, x_m, x_{m+1}, \dots, x_n]/(I_{\Delta_1}, I_{\Delta_2}, \{x_i x_j\}_{1 \le i \le m, m+1 \le j \le n}).$$

We claim that $\operatorname{depth}(K[x_1, \dots, x_m]/I_{\Delta_1}) > 0$ and $\operatorname{depth}(K[x_{m+1}, \dots, x_n]/I_{\Delta_2}) > 0$. Because if for example $\operatorname{depth}(K[x_1, \dots, x_m]/I_{\Delta_1}) = 0$, then $I_{\Delta_1} = (x_1, \dots, x_m)$. But, this implies that $\Delta_1 = \emptyset$ which contradicts our assumption on Δ_1. Now, the claim is immediate by Proposition 3.9. □

Theorem 3.11. *Let Δ be a locally complete intersection simplicial complex on $[n]$. Then Stanley's conjecture holds for S/I_Δ.*

Proof. If Δ is a connected, then Lemma 3.8 yields the claim. Otherwise, by [TY, Theorem 1.15], Δ is a finitely many disjoint union of non-empty simplicial complexes. So, in this case the assertion follows by Corollary 3.10. □

In [HP, Corollary 4.3] it is shown that if S/I is pretty clean, then it is sequentially Cohen-Macaulay. In [S1] this fact is reproved by a different argument and it is shown that depth of S/I is equal to the minimum of the dimension of $S/\mathfrak{p}$, where $\mathfrak{p} \in \operatorname{Ass}_S S/I$. This implies part a) of the following remark.

Remark 3.12. Let I be a monomial ideal of S and M a multigraded finitely generated S-module.

a) Assume that either:
   i) I is generated by a filter-regular sequence,
   ii) I is generated by a d-sequence,
   iii) I is almost complete intersection,
   iv) $\mu(I) \leq 3$; or
   v) I is the Stanley-Reisner ideal of a connected simplicial complex on $[n]$ which is locally complete intersection.

   Then both Stanley's and h-regularity conjectures hold for S/I. Also, in each of these cases S/I is sequentially Cohen-Macaulay and $\operatorname{depth} S/I = \min\{\dim S/\mathfrak{p} | \quad \mathfrak{p} \in \operatorname{Ass}_S S/I\}$.
b) We know that if S/I is almost clean, then Stanley's conjecture holds for S/I. By using Corollary 3.10, we can provide an example of a monomial ideal I of S such that Stanley's conjecture holds for S/I, while it is not almost clean. To this end, let Δ_1, Δ_2 and Δ be as in Corollary 3.10 and $\dim \Delta_i > 0$, $i = 1, 2$. Evidently, Δ is not shellable, and so [D, Theorem on page 53] implies that S/I_Δ is not almost clean. On the other hand, Stanley's conjecture holds for S/I_Δ by Corollary 3.10.

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